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Fusion Algebras & Accidental Trigonometry

Christopher D. Goff
University of the Pacific, cgoff@pacific.edu

Fusion Algebras & Accidental Trigonometry (or, How I Spent My Sabbatical)

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Outline

A Trig Problem

If You're Bored

Find a simple formula for

$$\alpha_\ell := \frac{1}{2}\prod_{j=1}^{\ell} 2\cos\left(\frac{2\pi(2j-1)}{8\ell}\right).$$

For example:

$$\alpha_1 = \frac{1}{2}\cdot 2\cos\left(\frac{2\pi}{8}\right) = \frac{1}{\sqrt{2}}.$$

Also,

$$\alpha_2 = \frac{1}{2}2\cos\left(\frac{2\pi}{16}\right)2\cos\left(\frac{6\pi}{16}\right) = \cos\left(\frac{8\pi}{16}\right) + \cos\left(\frac{4\pi}{16}\right) = \frac{1}{\sqrt{2}}.$$

Hint: $\alpha_\ell > 0$ for all ℓ.

What is a fusion algebra?

$\mathcal{F}$ is a fusion algebra if $\mathcal{F}$ is a commutative, associative ring with a (nice) finite $\mathbb{Z}_{\geq 0}$-basis that includes the identity element and is closed under a "dual" operation.

This means every element of $\mathcal{F}$ can be expressed as a $\mathbb{Z}_{\geq 0}$-linear combination of basis elements. In particular, the product of any two basis elements is a finite sum of basis elements (with non-negative integer coefficients).

The main, motivating example of a fusion algebra arises in the representation theory of finite groups over $\mathbb{C}$, but fusion algebras also arise in the representation theory of other algebraic objects.

Example: S_3 irreducibles

For us, $\mathcal{F}$ is a ring with $\mathbb{Z}$-basis consisting of self-dual $A = \mathrm{id}_{\mathcal{F}}$, B, and C, and with multiplication given by:

$$\begin{aligned} B \cdot B &= A \\ B \cdot C &= C \\ \& \quad C \cdot C &= A + B + C \end{aligned}$$

on the basis elements, and extended linearly. For instance:

$$(2A + C) \cdot C = 2(A \cdot C) + (C \cdot C) = 2C + A + B + C = A + B + 3C.$$

Example: S_3 irreducibles (cont.)

Notice that multiplication by an element is a linear operator. So, we can write down matrices for multiplication by A, B, and C with respect to $\{A, B, C\}$. For example, if M_C is multiplication by C, then

$$M_C(A) = C, \quad M_C(B) = C, \quad M_C(C) = A + B + C.$$

So,

$$[M_C] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Example: S_3 irreducibles (cont.)

Similarly,

$$[M_A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [M_B] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If You're Bored

Show that $[M_A]$, $[M_B]$, and $[M_C]$ obey the same multiplication rules as A, B, and C do, namely:

$$\begin{aligned} A &= \mathrm{id}_{\mathcal{F}} \\ B \cdot B &= A \\ B \cdot C &= C \\ \& \quad C \cdot C &= A + B + C. \end{aligned}$$

Cue from physicists

Now we have three commuting matrices.

Since the "duality" corresponds to the transpose, these matrices always commute with their transposes, meaning that they can be not only diagonalized, but by *orthogonal* eigenvectors.

Therefore, these three matrices can be *simultaneously* diagonalized by orthogonal eigenvectors.

If You're Bored

Find simultaneous eigenvectors for $[M_A]$, $[M_B]$, *and* $[M_C]$.

Spectra

- The eigenvalues of $[M_A]$ are 1, 1, and 1.
- The eigenvalues of $[M_B]$ are 1, -1, and 1.

$$0 = \det([M_C] - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \lambda(2-\lambda)(1+\lambda)$$

- The eigenvalues of $[M_C]$ are 2, 0, and -1.

Simultaneous Eigenvectors

In the interests of time:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a set of simultaneous (orthogonal) eigenvectors for $[M_A]$, $[M_B]$, and $[M_C]$.

Check:

$$
\begin{array}{c|ccc}
 & \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\
\hline
[M_A] & & & \\
[M_B] & & & \\
[M_C] & & &
\end{array}
$$

Check:

$$
\begin{array}{c|ccc}
 & \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\
\hline
[M_A] & 1 & 1 & 1 \\
[M_B] & 1 & -1 & 1 \\
[M_C] & 2 & 0 & -1
\end{array}
$$

Check:

$$\begin{array}{c|ccc} & \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ \hline [M_A] & 1 & 1 & 1 \\ [M_B] & 1 & -1 & 1 \\ [M_C] & 2 & 0 & -1 \end{array}$$

Key Idea

The components of the eigenvectors can be chosen to be equal to the corresponding eigenvalues.

Eigendata

So, the matrix $S_{\mathcal{F}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$ contains all the eigenvalues **and eigenvectors** of the corresponding multiplication matrices of the basis elements of $\mathcal{F}$.

If You're Bored

Show that the rows of $S_{\mathcal{F}}$ form a fusion algebra isomorphic to $\mathcal{F}$ under component-wise operations. For example:

$$[2 \quad 0 \quad -1]^2 = [4 \quad 0 \quad 1] = [1 \quad 1 \quad 1] + [1 \quad -1 \quad 1] + [2 \quad 0 \quad -1].$$

S-matrix

Key Idea

$$\boxed{\mathcal{F} \Longleftrightarrow S}$$

Indeed, given a fusion algebra $\mathcal{F}$, there is a matrix S whose columns are simultaneous (orthogonal) eigenvectors for the basis elements of $\mathcal{F}$ and whose entries are the eigenvalues for the corresponding multiplication matrices.

Conversely, given such a matrix S, there is a fusion algebra $\mathcal{F}$ determined by the rows of S under component-wise operations.

So here's what I did ...

What I did on my sabbatical

I examined fusion algebras from two different mathematical structures:

1. vertex operator algebra for a rank-1 even lattice ($\mathbb{Z}[\sqrt{2\ell}]$), and
2. affine Kac-Moody Lie algebra of type $D_\ell^{(1)}$ (fusion level 2).

Using the Kac-Peterson formula (for Kac-Moody Lie algebras),

$$S_{\Lambda,\Lambda'} = c \sum_{w \in W_0} \det(w) \exp\left(- \frac{2\pi i \langle \overline{\Lambda} + \overline{\rho} \mid w(\overline{\Lambda'} + \overline{\rho}) \rangle}{k + h^\vee} \right),$$

Michael Cuntz (Technische Universität, Kaiserslautern) and I calculated each entry of the S-matrix for (2) and then showed that the rows of S obey the known multiplication rules of (1). Therefore these two fusion algebras are **isomorphic**.

Using Euler: $\exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta$

Along the way, we found that some entries of S are determinants of matrices whose entries are sums of roots of unity.

Fix $\ell \in \mathbb{N}$. Notice $\exp\left(\frac{2\pi \mathrm{i} k}{\ell}\right) + \exp\left(-\frac{2\pi \mathrm{i} k}{\ell}\right) = 2\cos\left(\frac{2\pi k}{\ell}\right)$.

Two of the (ℓ by ℓ) matrices that arose in our work we called X and Ω, where

$$X_{ij} = 2\cos\left(\frac{2\pi(2i-1)(2j-1)}{8\ell}\right) \quad \text{and} \quad \Omega_{ij} = 2\cos\left(\frac{2\pi(i-1)(2j-1)}{4\ell}\right).$$

We needed to find $\frac{\det X}{\det \Omega}$.

Cosine Identities, I

If You're Bored

Using the following identity, show that $X^2 = (2\ell)I_\ell$ *and find* $\Omega\Omega^T$.

Nifty Trig

$$1 + 2\cos x + 2\cos 2x + \ldots + 2\cos \ell x = \frac{\sin\left(\frac{(2\ell+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

We can conclude that $(\det X)^2 = (2\ell)^\ell$ and $(\det \Omega)^2 = 2(2\ell)^\ell$. In other words,

$$\left(\frac{\det X}{\det \Omega}\right)^2 = \frac{1}{2}.$$

Silly minus signs

So, $\dfrac{\det X}{\det \Omega} = \pm\dfrac{1}{\sqrt{2}}$. But which is it, plus or minus? Does it depend on ℓ?

[There were a lot of minus signs around, and we had some leeway as to how we assigned the rows of S to the irreducibles of the VOA - we weren't too sure of any of the signs.]

Using a computer, we found that $\dfrac{\det X}{\det \Omega} = \dfrac{1}{\sqrt{2}}$ for small ℓ.

Lemma and Proof

Lemma (Cuntz-G, 2008-9)

For all ℓ, $\dfrac{\det X}{\det \Omega} = \dfrac{1}{\sqrt{2}}$.

Proof: First, let $\beta_j = \left(\dfrac{2\pi(2j-1)}{8\ell}\right)$. Then $X_{ij} = 2\cos\left((2i-1)\beta_j\right)$.
Divide column j of X by $2\cos\beta_j$, which gives a new matrix Y satisfying

$$\begin{aligned} Y_{ij} &= \frac{2\cos((2i-1)\beta_j)}{2\cos\beta_j} = \frac{\exp((2i-1)\beta_j\mathrm{i}) + \exp(-(2i-1)\beta_j\mathrm{i})}{\exp(\beta_j\mathrm{i}) + \exp(-\beta_j\mathrm{i})} \\ &\overset{(*)}{=} \exp((2i-2)\beta_j\mathrm{i}) - \exp((2i-4)\beta_j\mathrm{i}) + \dots \\ &\quad - \exp(-(2i-4)\beta_j\mathrm{i}) + \exp(-(2i-2)\beta_j\mathrm{i}) \\ &= 2\cos((2i-2)\beta_j) - 2\cos((2i-4)\beta_j) + \dots + (-1)^{i-1}. \end{aligned}$$

Aside: Cosine Identities, II

(These are specific examples of the calculations above.)

Nifty Trig

$$\frac{\cos(3x)}{\cos x} = 2\cos(2x) - 1$$

$$\frac{\cos(5x)}{\cos x} = 2\cos(4x) - 2\cos(2x) + 1$$

$$\frac{\cos(7x)}{\cos x} = 2\cos(6x) - 2\cos(4x) + 2\cos(2x) - 1$$

$$\frac{\cos(3x)}{\cos x} = \frac{4\cos^3(x) - 3\cos x}{\cos x} = 2(2\cos^2 x - 1) - 1$$

Proof (cont.)

Add row i of Y to row $i+1$ (telescoping!), which gives a new matrix Z satisfying

$$\begin{aligned}
Z_{1j} &= 1 = \frac{1}{2}(2\cos(0)) = \frac{1}{2}\Omega_{1j} \\
Z_{ij} &= 2\cos((2i-2)\beta_j) \\
&= 2\cos\left(\frac{2\pi(2i-2)(2j-1)}{8\ell}\right) \\
&= 2\cos\left(\frac{2\pi(i-1)(2j-1)}{4\ell}\right) = \Omega_{ij}
\end{aligned}$$

So

$$\det\Omega = 2\det Z = 2\det Y = \frac{2\det X}{\prod_{j=1}^{\ell} 2\cos\beta_j}.$$

Proof (cont.)

Hence,

$$\frac{\det X}{\det \Omega} = \frac{1}{2} \prod_{j=1}^{\ell} 2 \cos \beta_j.$$

Notice that this is exactly α_ℓ from the beginning of the talk.

Proof (cont.)

Hence,

$$\frac{\det X}{\det \Omega} = \frac{1}{2}\prod_{j=1}^{\ell} 2\cos\beta_j = \alpha_\ell.$$

So, since $\alpha_\ell > 0$,

$$\frac{\det X}{\det \Omega} = \frac{1}{\sqrt{2}}.$$

Cosine Identities, III

Nifty Trig

Therefore, for all ℓ,

$$\alpha_\ell = \frac{1}{2}\prod_{j=1}^{\ell} 2\cos\left(\frac{2\pi(2j-1)}{8\ell}\right) = \frac{1}{\sqrt{2}}.$$

In other words,

$$\frac{2^4}{2}\cos\left(\frac{2\pi}{32}\right)\cos\left(\frac{6\pi}{32}\right)\cos\left(\frac{10\pi}{32}\right)\cos\left(\frac{14\pi}{32}\right) = \frac{1}{\sqrt{2}},$$

etc.

Lessons Learned

- Fusion algebras are nice, and can be summarized in an S-matrix.
- Sometimes you need lots of different areas of math (abstract algebra, linear algebra, trigonometry) to solve a problem.
- You never know where your problems will lead.

Thank you!

And thanks to xkcd at http://www.xkcd.com/552/